Probabilistic Modeling on Rankings

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ACML, Singapore, November 4th, 2012
Outline of the talk

1. Introduction
2. Thurstone Models
3. Plackett-Luce Model
4. Ranking Induced by Pair-comparisons
5. Distance-Based Ranking Models
6. Other Models
7. Independence
8. Datasets and Software
9. Conclusions
Problems we are interested in

Identity-tracking
Probabilistic Modeling on Rankings

Introduction

Problems we are interested in

Preferences
Probabilistic Modeling on Rankings

Introduction

Problems we are interested in

Information retrieval
Remarks

- Social science
- Machine learning: ECML, ICML, UAI, NIPS, JMLR
- NOT “learning to rank”
- A model can be explained from many different points of view
- *Probabilistic graphical model bias*
What are permutations?

- A permutation $\sigma$ can be seen as a bijection between the set $\{1, 2, \ldots, n\}$ onto itself:

$$\sigma : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}$$

$$i \quad \mapsto \quad \sigma(i)$$

- Interpretation
  - $\sigma(i)$ represents the rank associated to the $i$-th element
  - $\sigma^{-1}(i)$ represents the $i$-th ranked element

- $S_n$ is the set of permutations of $n$ elements, then $(S_n, o)$ forms the symmetric group:

$$\sigma_1 \circ \sigma_2(i) = \sigma_1(\sigma_2(i))$$
Probabilistic Modeling on Rankings

Introduction

Permutations

Notation and representation

- The identity permutation \((1 \ 2 \ldots \ n)\) will be denoted by \(e\).
- A permutation \(\sigma\) can be equivalently represented as a permutation matrix \(M = [m_{ij}]\) where:

\[
m_{ij} = \begin{cases} 
1 & \text{if } \sigma(i) = j \\
0 & \text{otherwise}
\end{cases}
\]
Probabilistic Modeling on Rankings

Introduction

Learning probability distributions over permutations

\[ p : S_n \rightarrow [0, 1] \]

\[ \sigma \mapsto p(\sigma) \]

\[
\begin{align*}
(1 & 2 3 4 5 6 7) \\
(2 & 3 1 7 6 5 4) \\
(6 & 7 1 3 4 2 5) \\
(5 & 2 7 6 3 4 1) \\
(5 & 1 7 2 4 3 6) \\
(3 & 5 7 1 2 4 6) \\
\ldots
\end{align*}
\]
Learning probability distributions over permutations

\[ p : S_n \rightarrow [0, 1] \]

\[ \sigma \leftrightarrow p(\sigma) \]

(1 2 3 4 5 6 7 \ldots)
(2 3 1 7 6 5 4 \ldots)
(6 7 1 3 4 2 5 \ldots)
(5 2 7 6 3 4 1 \ldots)
(5 1 7 2 4 3 6 \ldots)
(3 5 7 1 2 4 6 \ldots)
\vdots
Learning probability distributions over permutations

\[ p : S_n \rightarrow [0, 1] \]
\[ \sigma \leftrightarrow p(\sigma) \]

1 \succ 2 \succ 4, 5
2, 3, 1 \succ 7 \succ 6, 5, 4
6, 7, 1, 3 \succ 4 \succ 2
5, 2 \succ 7, 6 \succ 3, 4 \succ 1
\vdots
Probabilistic Modeling on Rankings

Introduction

Learning probability distributions over permutations

\[
p : S_n \longrightarrow [0, 1]
\]

\[
\sigma \longmapsto p(\sigma)
\]

\[
(1\; 2\; 3\; 4 \quad \ldots \quad 5\; 6\; 7)
\]
\[
(2\; 3\; 1\; 7 \quad \ldots \quad 6\; 5\; 4)
\]
\[
(6\; 7\; 1\; 3 \quad \ldots \quad 4\; 2\; 5)
\]
\[
(5\; 2\; 7\; 6 \quad \ldots \quad 3\; 4\; 1)
\]
\[
(5\; 1\; 7\; 2 \quad \ldots \quad 4\; 3\; 6)
\]
\[
(3\; 5\; 7\; 1 \quad \ldots \quad 2\; 4\; 6)
\]
\[
\ldots
\]
Probabilistic Modeling on Rankings

Introduction

Learning probability distributions over permutations

\[ \begin{align*}
  1 & \sim 5 \\
  1 & \sim 7 \\
  3 & \sim 4 \\
  5 & \sim 7 \\
  \vdots
\end{align*} \]

\[ p : S_n \longrightarrow [0, 1] \]

\[ \sigma \longmapsto p(\sigma) \]
Probabilistic Modeling on Rankings

Introduction

Representing probability distributions over permutation

**Problems**

- How many parameters are needed?

\[ p(1\ 2\ 3\ 4\ 5) , \ p(2\ 1\ 3\ 4\ 5) \]
\[ p(3\ 2\ 1\ 4\ 5) , \ \ldots \]
\[ \ldots , \ p(5\ 4\ 3\ 2\ 1) \]

- \(5! - 1\) parameters. In general \(n! - 1\)
Probabilistic Modeling on Rankings

Introduction

Representing probability distributions over permutation

Problems

• How many parameters are needed?

\[ p(1 \ 2 \ 3 \ 4 \ 5) \ , \ p(2 \ 1 \ 3 \ 4 \ 5) \]
\[ p(3 \ 2 \ 1 \ 4 \ 5) \ , \ \ldots \]
\[ \ldots \ , \ p(5 \ 4 \ 3 \ 2 \ 1) \]

• 5! – 1 parameters. In general \( n! – 1 \)
Probabilistic Modeling on Rankings

Introduction

Bayesian networks

\[ p(x) = p(x_1) \cdot p(x_2) \cdot p(x_3|x_1, x_2) \cdot p(x_4|x_3) \cdot p(x_5|x_3, x_6) \cdot p(x_6) \]
Probabilistic Modeling on Rankings

Introduction

Bayesian networks

\[
p(x) = p(x_1) \cdot p(x_2) \cdot p(x_3|x_1, x_2) \cdot p(x_4|x_3) \cdot p(x_5|x_3, x_6) \cdot p(x_6)
\]
Bayesian networks

\[ \theta_4 = (\theta_{41}, \theta_{42}) \]

\[ p(x) = p(x_1) \cdot p(x_2) \cdot p(x_3 | x_1, x_2) \cdot p(x_4 | x_3) \cdot p(x_5 | x_3, x_6) \cdot p(x_6) \]
Probabilistic Modeling on Rankings

Introduction

Bayesian networks

$$\theta_4 = (\theta_{41}, \theta_{42})$$

$$\begin{cases} 
\theta_{411} = p(X_4 = 0 \mid X_3 = 0) \\
\theta_{412} = p(X_4 = 1 \mid X_3 = 0) 
\end{cases}$$

$$p(\mathbf{x}) = p(x_1) \cdot p(x_2) \cdot p(x_3 \mid x_1, x_2) \cdot p(x_4 \mid x_3) \cdot p(x_5 \mid x_3, x_6) \cdot p(x_6)$$
Probabilistic Modeling on Rankings

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Bayesian networks

\[ \theta_4 = (\theta_{41}, \theta_{42}) \]

\[ \begin{align*}
\theta_{411} &= p(X_4 = 0 \mid X_3 = 0) \\
\theta_{412} &= p(X_4 = 1 \mid X_3 = 0)
\end{align*} \]

General case

\[ \theta_{ijk} = p(X_i = x_i^k \mid Pa_i = pa_i^j) \]

\[ p(x) = p(x_1) \cdot p(x_2) \cdot p(x_3 \mid x_1, x_2) \cdot p(x_4 \mid x_3) \cdot p(x_5 \mid x_3, x_6) \cdot p(x_6) \]
Bayesian networks for permutations

\[ X_1 = 1 \Rightarrow X_2 \neq 1 \]
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Bayesian networks for permutations

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Bayesian networks for permutations

- $X_1 = 1 \Rightarrow X_2 \neq 1$
- $X_1 = 1 \Rightarrow X_3 \neq 1$
- $X_1 = 1 \Rightarrow X_4 \neq 1$
Bayesian networks for permutations

\[ X_1 = 1 \implies X_2 \neq 1 \]
\[ X_1 = 1 \implies X_3 \neq 1 \]
\[ X_1 = 1 \implies X_4 \neq 1 \]
Bayesian networks for permutations

\[ X_1 = 1 \Rightarrow X_2 \neq 1 \]

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\[ X_1 = 1 \Rightarrow X_4 \neq 1 \]
Probabilistic Modeling on Rankings

Introduction

Bayesian networks for permutations

\[ X_1 = 1 \Rightarrow X_2 \neq 1 \]
\[ X_1 = 1 \Rightarrow X_3 \neq 1 \]
\[ X_1 = 1 \Rightarrow X_4 \neq 1 \]

\[ p(x) = p(x_1) \cdot p(x_2|x_1) \cdot p(x_3|x_1, x_2) \cdot p(x_4|x_1, x_2, x_3) \]
Inferring over permutations

Recommender systems (Sun and Lebanon, 2012)

Which is the most preferred film given the current personal rankings?

$$\arg \max_{i \neq 3, 7, 9, 15, 4} p(\sigma^{-1}(1) = i \mid 3 \succ 7, 9, 15 \succ 4)$$

Which is the most probable order of the films given the current rankings?

$$\arg \max_{\sigma} p(\sigma \mid 3 \succ 7, 9, 15 \succ 4)$$
Inference over permutations

Information retrieval

Label ranking (Cheng and Hüllermeier, 2009; Chen et al., 2010): finding the ranking that maximizes the probability
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Thurstone Order Statistic Models

Ranking biscuits in relation with its sweetness

B1
B2
B3
B4
B5
Ranking biscuits in relation with its sweetness

B1 →
B2
B3
B4
B5
Thurstone Order Statistic Models

<table>
<thead>
<tr>
<th>B1</th>
<th>→</th>
<th>5.6</th>
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<tbody>
<tr>
<td>B2</td>
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<table>
<thead>
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<th>Sweetness</th>
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<tr>
<td>B1</td>
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Thurstone Order Statistic Models

Ranking biscuits in relation with its sweetness

B1  →  5.6
B2  →  4.9
B3  →  9.3
B4  →  2.1
B5  →  3.4

(__, __, __, __, __)
### Thurstone Order Statistic Models

#### Ranking biscuits in relation with its sweetness

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( _, _, _, _, _ )
## Thurstone Order Statistic Models

### Ranking biscuits in relation with its sweetness

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## Thurstone Order Statistic Models

Ranking biscuits in relation with its sweetness

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Thurstone Order Statistic Models

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Biscuits ranked in relation to sweetness:

B1, B2, B3, B4, B5

(1, _, _, _)
Thurstone Order Statistic Models

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Ranking biscuits in relation with its sweetness
## Thurstone Order Statistic Models

 Ranking biscuits in relation with its sweetness

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( _, _, _, 1, 2 )
### Thurstone Order Statistic Models

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(4, 3, 5, 1, 2)
Probabilistic Modeling on Rankings
Thurstone Models

Thurstone Order Statistic Models

<table>
<thead>
<tr>
<th>Biscuit</th>
<th>Score</th>
<th>~</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>B1</td>
<td>5.6</td>
<td>~</td>
<td>X₁</td>
</tr>
<tr>
<td>B2</td>
<td>4.9</td>
<td>~</td>
<td>X₂</td>
</tr>
<tr>
<td>B3</td>
<td>9.3</td>
<td>~</td>
<td>X₃</td>
</tr>
<tr>
<td>B4</td>
<td>2.1</td>
<td>~</td>
<td>X₄</td>
</tr>
<tr>
<td>B5</td>
<td>3.4</td>
<td>~</td>
<td>X₅</td>
</tr>
</tbody>
</table>

Ranking biscuits in relation with its sweetness

(4, 3, 5, 1, 2)
Thurstone Order Statistic Models

Basics

- Each item is associated with a true continue value: sweetness of a cookie, loudness of a sound, etc.
- A judge assesses the cookies or the sounds and classifies them.
- Errors are produced because of the lack of exactness of the sensorial apparatus of the judge.
- The output of the assessment is a classification of the items.
Basics

- Codify a permutation as a real-valued vector (random keys).
- Given a real vector \((x_1, x_2, \ldots, x_n)\) of length \(n\), a permutation can be obtained by ranking the positions using the values \(x_i \ (i = 1, \ldots, n)\).
- Given

\[
(2.35, 3.42, 9.35, 0.32, 11.54, 10.42, 5.23, 4.2, 7.8)
\]

the permutation obtained is \((2 \ 3 \ 7 \ 1 \ 9 \ 8 \ 5 \ 4 \ 6)\)
Probabilistic Modeling on Rankings
Thurstone Models

Thurstone Order Statistic Models

Definition

Given \( \{X_1, X_2, \ldots, X_n\} \) random variables with a continuous joint distribution \( F(x_1, \ldots, x_n) \), we can define a random ranking \( \sigma \) in such a way that \( \sigma(i) \) is the rank that \( X_i \) occupied between \( X_1, X_2, \ldots, X_n \). In this way:

\[
P(\sigma) = P(X_{\sigma^{-1}(1)} < X_{\sigma^{-1}(2)} < \ldots < X_{\sigma^{-1}(n)})
\]

Unconstrained \( X_i \)'s (and therefore \( F \)) produce all kind of distributions over \( S_n \)
Thurstone Models

n-1 parameter model

- Assumption: All the $X_i$ are independent (it produces a proper subset of the distributions over permutations)
- All the distributions are similar $f_i(x) = f(x - \mu_i)$
- The parameters of the model are $(\mu_2 - \mu_1, \ldots, \mu_n - \mu_1)$
- Most common models: $f$ Gaussian or $f$ Gumbel (Luce’s Model)
MLE of the parameters: complex numerical integral problem:

\[
p(\sigma) = P(X_{\sigma^{-1}(1)} < X_{\sigma^{-1}(2)} < \ldots < X_{\sigma^{-1}(n)})
\]
\[
= \int_{\Omega} f(x_1, x_2, \ldots, x_n) \, dx_1 \cdots dx_n
\]

where \( \Omega = \{(x_1, x_2, \ldots, x_n) \mid x_{\sigma^{-1}(1)} < x_{\sigma^{-1}(2)} < \ldots < x_{\sigma^{-1}(n)} \text{ with } x_i \in \mathcal{R} \quad i = 1, \ldots, n \} \)
<table>
<thead>
<tr>
<th>Learning Approaches</th>
</tr>
</thead>
<tbody>
<tr>
<td>Böckenholt (1993): Numerical integration ((n \leq 4))</td>
</tr>
<tr>
<td>Yao &amp; Böckenholt (1999): Bayesian Gibbs sampling ((n \leq 10))</td>
</tr>
<tr>
<td>Limited information (Maydeu-Olivares, 1999; 2001; 2003): pair, triplets and tetrads comparisons frequencies ((n \leq 10))</td>
</tr>
</tbody>
</table>
Thurstone Models

Sampling

- It depends on the complexity of $F(x_1, \ldots, x_n)$

TrueSkill$^TM$ (Herbrich et al. 2007)
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Plackett-Luce Model

# Plackett-Luce Model

## Ranking objects with Plackett-Luce

<table>
<thead>
<tr>
<th>O1</th>
</tr>
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<tbody>
<tr>
<td>O2</td>
</tr>
<tr>
<td>O3</td>
</tr>
<tr>
<td>O4</td>
</tr>
<tr>
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</tr>
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Plackett-Luce Model

Ranking objects with Plackett-Luce

O1
O2
O3
O4
O5
Probabilistic Modeling on Rankings

Plackett-Luce Model

**Plackett-Luce Model**

Ranking objects with Plackett-Luce

O1 $\rightarrow w_1$
O2
O3
O4
O5
Probabilistic Modeling on Rankings
Plackett-Luce Model

Plackett-Luce Model

Ranking objects with Plackett-Luce

O1 $\rightarrow w_1$
O2 $\rightarrow w_2$
O3 $\rightarrow w_3$
O4 $\rightarrow w_4$
O5 $\rightarrow w_5$
Probabilistic Modeling on Rankings

Plackett-Luce Model

Plackett-Luce Model

Ranking objects with Plackett-Luce

\[ O_1 \rightarrow w_1 \quad \frac{w_1}{w_1 + w_2 + w_3 + w_4 + w_5} \]

\[ O_2 \rightarrow w_2 \]

\[ O_3 \rightarrow w_3 \]

\[ O_4 \rightarrow w_4 \]

\[ O_5 \rightarrow w_5 \]
Probabilistic Modeling on Rankings
Plackett-Luce Model

Plackett-Luce Model

<table>
<thead>
<tr>
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<tr>
<td>O1 → ( w_1 ) ( \frac{w_1}{w_1 + w_2 + w_3 + w_4 + w_5} )</td>
</tr>
<tr>
<td>O2 → ( w_2 ) ( \frac{w_2}{w_1 + w_2 + w_3 + w_4 + w_5} )</td>
</tr>
<tr>
<td>O3 → ( w_3 ) ( \frac{w_3}{w_1 + w_2 + w_3 + w_4 + w_5} )</td>
</tr>
<tr>
<td>O4 → ( w_4 ) ( \frac{w_4}{w_1 + w_2 + w_3 + w_4 + w_5} )</td>
</tr>
<tr>
<td>O5 → ( w_5 ) ( \frac{w_5}{w_1 + w_2 + w_3 + w_4 + w_5} )</td>
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## Plackett-Luce Model

### Ranking objects with Plackett-Luce

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<th>Probability</th>
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</thead>
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<tr>
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<td>$\frac{w_1}{w_1 + w_2 + w_3 + w_4 + w_5}$</td>
</tr>
<tr>
<td>O2</td>
<td>$\frac{w_2}{w_1 + w_2 + w_3 + w_4 + w_5}$</td>
</tr>
<tr>
<td>O3</td>
<td>$\frac{w_3}{w_1 + w_2 + w_3 + w_4 + w_5}$</td>
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# Plackett-Luce Model

## Ranking objects with Plackett-Luce

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<tr>
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<th>Weight</th>
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<tr>
<td>O1</td>
<td>$w_1$</td>
<td>( \frac{w_1}{w_1 + w_2 + w_3 + w_4 + w_5} )</td>
</tr>
<tr>
<td>O2</td>
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<td>( \frac{w_2}{w_1 + w_2 + w_3 + w_4 + w_5} )</td>
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</tr>
<tr>
<td>O4</td>
<td>$w_4$</td>
<td>( \frac{w_4}{w_1 + w_2 + w_3 + w_4 + w_5} )</td>
</tr>
<tr>
<td>O5</td>
<td>$w_5$</td>
<td>( \frac{w_5}{w_1 + w_2 + w_3 + w_4 + w_5} )</td>
</tr>
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_(__, __, __, __, __, __)_
Probabilistic Modeling on Rankings

Plackett-Luce Model

## Plackett-Luce Model

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(, , , 1, )
### Plackett-Luce Model

#### Ranking objects with Plackett-Luce

<table>
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<th>Object</th>
<th>Probability</th>
<th>Formula</th>
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| O1     | $w_1$       | \[
\frac{w_1}{w_1 + w_2 + w_3 + w_4 + w_5}
\] |
| O2     | $w_2$       | \[
\frac{w_2}{w_1 + w_2 + w_3 + w_4 + w_5}
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(\_,\_,\_,\_1,\_)}
Probabilistic Modeling on Rankings
Plackett-Luce Model

Plackett-Luce Model

Ranking objects with Plackett-Luce

\[ \begin{align*}
O_1 & \rightarrow w_1 & \frac{w_1}{w_1 + w_2 + w_3 + w_4 + w_5} \\
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Plackett-Luce Model

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### Plackett-Luce Model

**Probabilistic Modeling on Rankings**

**Plackett-Luce Model**

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*Note: The ranking order for objects is denoted by $(_, _, _, 1, 2)$.*
Plackett-Luce Model

Ranking objects with Plackett-Luce

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Plackett-Luce Model

Ranking objects with Plackett-Luce

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Plackett-Luce Model

Ranking objects with Plackett-Luce

\( O_1 \rightarrow w_1 \) : \( \frac{w_1}{w_1 + w_2 + w_3 + w_4 + w_5} \)

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\( (4, 3, 5, 1, 2) \)
Probabilistic Modeling on Rankings

Plackett-Luce Model

Plackett-Luce Model

Ranking objects with Plackett-Luce

\[ O1 \rightarrow w_1 \quad \frac{w_1}{w_1 + w_2 + w_3 + w_4 + w_5} \]

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\[ p(4, 3, 5, 1, 2) = \frac{w_4}{w_1 + w_2 + w_3 + w_4 + w_5} \cdot \frac{w_5}{w_1 + w_2 + w_3 + w_4 + w_5} \cdot \frac{w_2}{w_1 + w_2 + w_3 + w_4 + w_5} \cdot \frac{w_1}{w_1 + w_2 + w_3 + w_4 + w_5} \]
Plackett-Luce Model

Definition procedure

- Parameters: $W = (w_1, w_2, \ldots, w_n)$ with $\sum_{i=1}^{n} w_i = 1$ and $w_i > 0$
- $w_i$ is the probability of chosen object $i$
- Procedure:
  1. An object is chosen using $W$
  2. Update $W$ and chose another object

Model

$$P(\sigma) = \prod_{i=1}^{n-1} \frac{W_{\sigma^{-1}(i)}}{\sum_{j=i}^{n} W_{\sigma^{-1}(j)}}$$
Plackett-Luce Model

Properties

- The choice probability ratio between two items is independent of any other items in the set.
- Easy to extend to partial rankings:
  \[ P(\sigma) = \prod_{i=1}^{n'-1} \frac{w_{\sigma^{-1}(i)}}{\sum_{j=i}^{n'} w_{\sigma^{-1}(j)}} \]
- Plackett-Luce model can be seen as a Thurstone model with f Gumbel.
Plackett-Luce model

**Learning**

- Given a dataset \( \{\sigma^1, \sigma^2, \ldots, \sigma^N\} \) find \( W \)
- Two main approaches have been developed
  - MM algorithm (Hunter, 2004)
  - Bayesian approach by means of message passing (Guiver and Snelson, 2009)
Minorization-Maximization (MM) algorithm

- It is an iterative algorithm to calculate ML parameters
- EM algorithm can be considered as a special case of MM
- The key idea: Find a function $Q(w|w')$ such that
  (minorization):
  $$Q(w|w') \leq L(w)$$

  with equality if $w = w'$
- In this case it happens that:
  $$\text{if } Q(w|w') \geq Q(w'|w') \text{ then } L(w) \geq L(w')$$
P-L Learning

**Minorization-Maximization (MM) algorithm**

The algorithm is as follows:

1. Give an initial guess for the parameters $w^0$
2. Set $l = 0$
3. Construct function $Q(w|w')$
4. Find $w^{l+1} = \text{argmax}_w Q(w|w')$
5. If $\|w^{l+1} - w^l\| < \theta$ stop. Otherwise set $l = l + 1$ and go to step 3
Probabilistic Modeling on Rankings
Plackett-Luce Model

P-L Learning

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How to construct $Q(w|w')$
Plackett-Luce Model

P-L Learning

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How to construct $Q(w|w^l)$

How to calculate $\arg\max_w Q(w|w^l)$
Probabilistic Modeling on Rankings
Plackett-Luce Model

P-L Learning

MM for P-L

Maximum Likelihood function given a sample \( \{\sigma^1, \ldots, \sigma^N\} \):

\[
\ln L(w | \sigma^1, \ldots, \sigma^N) = \sum_{k=1}^{N} \sum_{i=1}^{n-1} \left( \ln w_{(\sigma^k)^{-1}(i)} - \ln \sum_{j=i}^{n} w_{(\sigma^k)^{-1}(j)} \right)
\]

Constructing \( Q \)

\( \forall x, y > 0 \) we have \(- \ln x \geq 1 - \ln y - \frac{x}{y} \)

Taking \( y \) in the previous formula as \( w^l \) and applying it to \( L \) we obtain \( Q \)
P-L Learning

MM for P-L. Function $Q$

$$Q(w|w') = \sum_{k=1}^{N} \sum_{i=1}^{n-1} \left( \ln w_{(\sigma^k)^{-1}(i)} - \frac{\sum_{j=i}^{n} w_{(\sigma^k)^{-1}(j)}}{\sum_{j=i}^{n} w'_l(\sigma^k)^{-1}(j)} \right)$$
Probabilistic Modeling on Rankings

Plackett-Luce Model

P-L Learning

MM for P-L. Maximize Q

We maximize $Q$ by deriving and equating to 0:

$$w_r^{l+1} = \frac{h_r}{\sum_{k=1}^{N} \sum_{i=1}^{n-1} \delta_{kir} \left[ \sum_{j=i}^{n} w_{(\sigma_k)^{-1}(j)}^{l} \right]^{-1}}$$

where $h_r$ is the number of rankings in which the $r$-th item is ranked higher than last and

$$\delta_{kir} = \begin{cases} 1 & \text{if an item higher than } i - 1 \text{ is ranked } r \\ 0 & \text{otherwise} \end{cases}$$

in the $k$-th permutation.
Comments on the MM approach

- To guarantee convergence several properties have to be complied with. For instance:

  ...in every possible partition of the items into two nonempty subsets, some item in the second set ranks higher than some item in the first set at least once..

- May be applied to partial rankings without any changes
- May be combined with Newton-Raphson method
- Hunter (2004) reports experiments with up to $n = 80$
Probabilistic Modeling on Rankings
Plackett-Luce Model

P-L Learning

Bayesian Approach (Guiver and Snelson, 2009)

- Avoid the overfitting that ML parameters can suffer in certain situations
- Give a method that may be globally applied without the constraints of the MM method
- Bayesian approach: given the dataset, assume a priori distribution over the parameters and carry out the inference that will obtain the a posteriori distribution
### Bayesian Approach (Guiver and Snelson, 2009)

- The authors assume a Gamma distribution as a prior:
  \[ w_i \sim \text{Gamma}(v \mid \alpha_0, \beta_0) \]

- Assume a full factorization for the posterior
  \[ p(w) \approx \prod_{i=1}^{n} p(w_i) \]

- Use a message-passing algorithm (power Expectation-Propagation) to carry out inference
P-L Inference

- Sampling is trivial in this model
- Marginal calculation can be exponential: $P(\sigma^{-1}(n) = i)$
Machine Learning Applications

ML Applications

- Cheng et al. (2010): Label ranking
- Chen et al. (2007): Document ranking
Probabilistic Modeling on Rankings
Ranking Induced by Pair-comparisons

Outline of the talk

1. Introduction
2. Thurstone Models
3. Plackett-Luce Model
4. Ranking Induced by Pair-comparisons
5. Distance-Based Ranking Models
6. Other Models
7. Independence
8. Datasets and Software
9. Conclusions
Babington-Smith proposes a model based on pair comparisons:

\[ p_{i,j} = \text{probability of preferring item } i \text{ to item } j \]

if only that comparison were to be made

Assuming no ties, for \( n \) objects there are \( \binom{n}{2} \) possible comparisons
Ranking induced by pair-comparisons

Definition

- Given an ordering it is easy to get the pair comparisons:
  \((312) \Rightarrow 3 > 1, \ 3 > 2, \ 1 > 2\)

- The opposite is not true in general:
  \(3 > 1, \ 1 > 2, \ 2 > 3\)
A ranking is obtained carrying out all kind of comparisons until a consistent set of comparisons is obtained:

\[ P(\sigma) \propto \prod_{(i,j) | \sigma^{-1}(i) < \sigma^{-1}(j)} p_{i,j} \]

For instance given the ordering (1 3 4 2):

\[ p(1 3 4 2) \propto p_{1,3} \cdot p_{1,4} \cdot p_{1,2} \cdot p_{3,4} \cdot p_{3,2} \cdot p_{4,2} \]

The number of parameters is quadratic \( n(n-1)/2 \)

There is no closed form for the normalization constant

Simplifications....
Ranking induced by pair-comparisons

Mallows-Bradley-Terry models

- The model comes defined by a vector of weights $(v_1, \ldots, v_n)$ each one associated with an item:

\[ p_{i,j} = \frac{v_i}{v_i + v_j} \quad \text{with} \quad \sum_{i=1}^{n} v_i = 1 \quad \text{and} \quad v_i > 0 \]

- MM algorithms have been given to learn the MLE parameters (Hunter, 2004)
Probabilistic Modeling on Rankings
Ranking Induced by Pair-comparisons

Ranking induced by pair-comparisons

Extensions to Mallows-Bradley-Terry models

- Agresti (1990) considers a model where the probability of $i$ beating $j$ depends on which individual is listed first:

$$p_{ij} = \begin{cases} 
\frac{\theta w_i}{(\theta v_i + v_j)} & \text{if } i \text{ is home} \\
\frac{v_i}{(v_i + \theta v_j)} & \text{if } j \text{ is home}
\end{cases}$$

- Rao and Kupper (1967) allow ties:

$$P(i \text{ beats } j) = \frac{v_i}{(v_i + \theta v_j)}$$

$$P(j \text{ beats } i) = \frac{v_j}{(\theta v_i + v_j)}$$

$$P(j \text{ ties } i) = \frac{(\theta^2 - 1)v_i v_j}{((\theta v_i + v_j)(v_i + \theta v_j))}$$
Outline of the talk

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2. Thurstone Models
3. Plackett-Luce Model
4. Ranking Induced by Pair-comparisons
5. **Distance-Based Ranking Models**
6. Other Models
7. Independence
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9. Conclusions
Distance-based ranking models (Mallows models)

Definition

\[ p(\sigma | \theta, \sigma_0) = \frac{1}{Z(\theta)} e^{-\theta d(\sigma, \sigma_0)} \]

- It is an exponential model
- \( d \) is a distance between permutations such that:
  - \( d(\sigma, \pi) \geq 0 \) \( \forall \sigma, \pi \) with equality iff \( \sigma = \pi \)
  - Right-invariant property: \( d(\sigma, \pi) = d(\sigma \phi, \pi \phi) \) \( \forall \sigma, \pi, \phi \)
- Two parameters:
  - Central permutation \( \sigma_0 \)
  - Spread parameter \( \theta \geq 0 \)
- \( Z(\theta) \) is the partition function
Mallows model

Equivalent to Gaussian distribution for permutations
Mallows model

Equivalent to Gaussian distribution for permutations
Distances between permutations I

- **Kendall-tau metric.** Measures the number of disagreements between two permutations:

\[
T(\pi, \sigma) = \sum_{i<j} I\{ (\pi(i) - \pi(j))(\sigma(i) - \sigma(j)) < 0 \}
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Distances between permutations

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**Example**

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Probabilistic Modeling on Rankings
Distance-Based Ranking Models

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\[ T((2\ 3\ 1\ 5\ 4), (1\ 3\ 4\ 2\ 5)) = 5 \]
Distances between permutations II

- Kendall-tau is equivalent to: minimum number of adjacent swaps to go from $\pi^{-1}$ to $\sigma^{-1}$.
- $0 \leq T(\pi, \sigma) \leq n(n - 1)/2$
Distances between permutations III

- Spearman rho metric:
  \[ R(\pi, \sigma) = \left( \sum_{i=1}^{n} (\pi(i) - \sigma(i))^2 \right)^{1/2} \]

- Spearman footrule:
  \[ F(\pi, \sigma) = \sum_{i=1}^{n} |\pi(i) - \sigma(i)| \]
Distances between permutations IV

- **Hamming distance:**
  \[ H(\pi, \sigma) = \sum_{i=1}^{n} I\{\pi(i) \neq \sigma(i)\} \]

- **Cayley metric:**
  \[ C(\pi, \sigma) = \text{minimum number of transpositions to go from } \pi \text{ to } \sigma \]
Distances between permutations $V$

- Ulam metric

\[ U(\pi, \sigma) = n - \text{maximum number of items ranked the same relative order by } \pi \text{ and } \sigma \]

- Example:

\[ e = (1 \ 2 \ 3 \ 4 \ 5) \quad \pi = (1 \ 4 \ 3 \ 2 \ 5) \]
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\[ U(e, \pi) = 5 - 3 = 2 \]
Probabilistic Modeling on Rankings
Distance-Based Ranking Models

Distances between permutations V

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- It is equivalent to the minimum number of “delete-shift-insert” operations to go from $\pi$ to $\sigma$
Probabilistic Modeling on Rankings
Distance-Based Ranking Models

## Mallows model

### Learning
- Given a dataset \( \{\sigma^1, \sigma^2, \ldots, \sigma^N\} \) find \( \sigma_0 \) and \( \theta \)
- Maximum likelihood estimation
  - \( \hat{\sigma}_0 = \arg\min_{\sigma} \sum_{i=1}^{N} d(\sigma^i, \sigma) \) (consensus ranking, NP-hard for \( T \))
  - \( \hat{\theta} \) can be found by a standard numerical method for convex optimization, given \( \hat{\sigma}_0 \)

### Inference
- For all distance \( d \), Gibbs sampling
- There are alternative methods depending on \( d \)
Mallows model with Kendall distance

Definition

- In this particular case the partition function can be calculated in a closed form:

\[
p(\sigma | \theta, \sigma_0) = \frac{1}{Z(\theta)} e^{-\theta d(\sigma, \sigma_0)}
\]

\[
p(\sigma | \theta, \sigma_0) = \frac{(1 - e^{-\theta})^{n-1}}{\prod_{i=1}^{n-1} (1 - e^{-(n-i+1)\theta})} e^{-\theta T(\sigma, \sigma_0)}
\]

- Learning. Kemeny ranking problem:

\[
\hat{\sigma}_0 = \arg \min_{\sigma} \sum_{i=1}^{N} T(\sigma_i, \sigma)
\]
Mallows model with Kendall distance

Learning

Ali and Meila (2012) compare 104 algorithms in the Kemeny ranking problem. Borda one of the best:

1. Calculate for all index $i$:

$$v_i = \frac{1}{N} \sum_{j=1}^{N} \sigma^j(i)$$

2. Assign $\hat{\delta}_0^{-1}(1)$ with the index of the smallest \{v_1, \ldots, v_n\}, $\hat{\delta}_0^{-1}(2)$ with second smallest and so on.
Mallows model

Problems

- Unimodality
- Estimation
- Permutations at the same distance from \( \sigma_0 \) have the same probability
Mallows model

Extensions

- Mixture of Mallows models (Murphy and Martin, 2003; Lee and Yu, 2012; Lu and Boutilier, 2011)
- Two-side infinite extension (Gnedin and Grigori Olshanski, 2012)
- Bao and Meila (2010) extends the Mallows model to infinite items. They learn the model from top-t lists of items and extend it to multi-modal data
- Generalized Mallows models
Generalized Mallows models

**Definition**

- Fligner and Verducci (1986): if a distance $d$ can be written as:

  $$d(\pi, \sigma) = \sum_{i=1}^{n} S_i(\sigma, \pi)$$

  and the $S_i$ are independent under the uniform distribution, then the Mallows model is **factorizable**. Even more, it can be generalized to a $n$-parameters model:

  $$P(\sigma) \propto \exp(-\sum_{j=1}^{n} \theta_j S_j(\sigma, \sigma_0))$$

  where a different spread parameter $\theta_i$ is associated with each position in the permutation.
**Generalized Mallows models**

**Kendall distance**

- Kendall distance can be decomposed as:

\[
T(\pi, \sigma) = \sum_{i=1}^{n-1} V_i(\pi, \sigma)
\]

where \( V_i(\pi, \sigma) \) is defined as follows:

\[
\sum_{j=i+1}^{n} I\{(\pi(i) - \pi(j))(\sigma(i) - \sigma(j)) < 0\}
\]

- Taking \( \pi = e \) there exists a bijection between the set of vectors \((v_1, \ldots, v_{n-1})\) and the set of permutations \(\sigma\)
Generalized Mallows models

Kendall distance

If we assign a different spread parameter $\theta_i$ to each position $i$ then

$$T(\sigma, \sigma_0) = \sum_{i=1}^{n-1} \theta_i V_i(\sigma, \sigma_0)$$

and the model can be written as:

$$p(\sigma) = \prod_{i=1}^{n-1} \frac{1}{\psi_i(\theta_i)} e^{-\theta_i V_i(\sigma, \sigma_0)} = \prod_{i=1}^{n-1} \frac{1 - e^{-\theta_i}}{1 - e^{-(n-i+1)\theta_i}} e^{-\theta_i V_i(\sigma, \sigma_0)}$$
### Generalized Mallows models with Kendall distance

#### Sampling
- The distribution of $V_i (i = 1, \ldots, n - 1)$ under the model is known:

  $$p(V_i(\sigma, \sigma_0) = r) = \frac{e^{-r\theta_i}}{\psi(\theta_i)}$$

#### Learning
Generalized Mallows model

Cayley distance

Define the following $X_i$ variables:

$$cycle_\sigma(i) = \{\sigma^k(i) | k = 0, 1, \ldots\}$$

where $\sigma^0(i) = i$ and $\sigma^k(i) = \sigma^{k-1}(\sigma(i))$

$$X_i(\sigma) = \begin{cases} 
0 & \text{if } i = \max\{cycle_\sigma(i)\} \\
1 & \text{otherwise}
\end{cases}$$
Generalized Mallows model

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- Example

$$\sigma = (3 \ 1 \ 2 \ 5 \ 4)$$
Generalized Mallows model

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2. Example

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Generalized Mallows model

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- Example

$\sigma = (3 \ 1 \ 2 \ 5 \ 4)$  
1 $\rightarrow$ 2 $\rightarrow$ 3 $\rightarrow$ 1  
4 $\rightarrow$ 5 $\rightarrow$ 4

$\text{cycle}_\sigma(1) = \text{cycle}_\sigma(2) = \text{cycle}_\sigma(3) = \{1, 2, 3\}$

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Generalized Mallows model

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\[X_1(\sigma) = X_2(\sigma) = 1 \quad X_3(\sigma) = 0 \quad X_4(\sigma) = 1 \quad X_5(\sigma) = 0\]
Generalized Mallows model

Cayley distance

- Cayley distance can be written as:

\[ C(\pi, \sigma) = C(\pi \sigma^{-1}, \sigma \sigma^{-1}) = C(\pi \sigma^{-1}, e) = \sum_{i=1}^{n-1} X_i(\pi \sigma^{-1}) \]

- A similar development that with Kendall can be given

  - A key difference: there is not a bijection between the set of binary vectors \((X_1, \ldots, X_{n-1})\) and the set of permutations \(S_n\)
Multi-stage ranking models

Definition

- The ranking is constructed in a process on $n - 1$ stages.
- At each step a decision is taken.
- A modal ranking $\sigma_0$ is assumed to exist.
- The decision only depends on the number of items left (the stage).
Multi-stage ranking models

\[ \sigma_0 = (2 \ 4 \ 3 \ 5 \ 1) \quad (\_ \_ \_ \_ \_ \_ \_ ) \]

\[ \sigma_0^{-1} = (5 \ 1 \ 3 \ 2 \ 4) \]
Multi-stage ranking models

\[ \sigma_0 = (2 \ 4 \ 3 \ 5 \ 1) \quad (\ - \ - \ - \ - \ - \ - ) \]
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Probability of each choice at stage 1:

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Multi-stage ranking models

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Multi-stage ranking models

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Probability of each choice at stage 2:

\{0.35, 0.3, 0.2, 0.15\}
Multi-stage ranking models

\[ \sigma_0 = (2 \ 4 \ 3 \ 5 \ 1) \quad (1 \ - \ - \ - \ - \ -) \]
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Multi-stage ranking models

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Probability of each choice at stage 2:

\{0.35, \ 0.3, \ 0.2, \ 0.15\}
Multi-stage ranking models

Definition

- It has $n(n - 1)/2$ parameters:
  
  $$p(m, r) = \text{probability of taken the } m\text{-th best decision at stage } r$$

- Let $V_r = m$ if the $(m + 1)$-th best decision is taken at stage $r$. Then
  
  $$P(\pi) = \prod_{r=1}^{n-1} p(V_r, r)$$

- Mallows and Plackett-Luce can be seen as multi-stage models
Outline of the talk

1. Introduction
2. Thurstone Models
3. Plackett-Luce Model
4. Ranking Induced by Pair-comparisons
5. Distance-Based Ranking Models
6. Other Models
7. Independence
8. Datasets and Software
9. Conclusions
First-order marginals

First-order marginals $Q = [q_{ij}]$:

$$q_{ij} = P(\sigma(i) = j)$$

- Easy to learn and to represent
- Which distribution is represented by these marginals?
Probabilistic Modeling on Rankings
Other Models

Models based on marginals

First-order marginals

- First-order marginals $Q = [q_{ij}]$:
  
  $$q_{ij} = P(\sigma(i) = j)$$

- Easy to learn and to represent
- Which distribution is represented by these marginals?
- Infinite distributions
Models based on marginals

First-order marginals

- Criterion to choose between them: The one with the maximum entropy

\[ P(\sigma) = \exp \left( \sum_{i=1}^{n} Y_{(i,\sigma(i))} - 1 \right) \]

where \( Y \in \mathbb{R}^{n \times n} \)

- Obtaining \( Y \) is \#P-hard (Agrawal et al., 2008)
### Models based on marginals

**High-order marginals**

- Huang et al. (2009) consider models based on high-order marginals
- They use the Fourier transform over the symmetric group to carry out inference tasks
- Irurozki et al. (2011) give a first approach to learning these kind of models
Non-parametric models I

Based on Mallows kernels

- Lebanon and Mao (2008) give a probabilistic approach based on Mallows kernels with Kendall distance

\[ p(\sigma) = \frac{1}{NZ(\theta)} \sum_{i=1}^{N} e^{-\theta T(\sigma, \sigma^i)} \]

- They were able to learn from partially ranked data:
Probabilistic Modeling on Rankings
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\[ \sigma = (2 3 1 7 6 5 4 \ldots) \]
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\[ \sigma = (2 \ 3 \ 1 \ 7 \ 6 \ 5 \ 4 \ - \ - \ - \ - \ \ldots) \]

\[ C_\sigma = \{ \pi | \pi(i) = \sigma(i), i = 1, 2, \ldots, 7 \} \]
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p(\sigma) = \frac{1}{NZ(\theta)} \sum_{i=1}^{N} \frac{1}{|C_{\sigma^i}|} \sum_{\tau \in C_{\sigma^i}} e^{-\theta T(\sigma, \tau)}
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\]
Non-parametric models II

Based on Mallows kernels

- Efficient learning
- Lacks: Marginalization, conditioning
### Non-parametric models III

#### Modifying Mallows kernels
- Sun and Lebanon (2012) use a triangular kernel based on Kendall distance to solve problems in recommendation systems.
- They can deal with all kind of partial rankings.

#### Based on the Fourier transform
- Barbosa y Kondor (2010) give another kernel approach based on the use of the Fourier transform.
Probabilistic Modeling on Rankings

Independence

Outline of the talk

1. Introduction
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3. Plackett-Luce Model
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9. Conclusions
Full independence - Definition (Huang et al. 2009)

- Independence based on relative rankings
- The subsets $A \subseteq \{1, \ldots, n\}$ and $B = A^c$ are independent under distribution $p(\sigma)$ if

$$p(\sigma) = p_A(\sigma_A) \cdot p_B(\sigma_B)$$

- Difficult to hold in practice

Full independence - Necessary condition

Each subset of the items maps to a particular subset of the positions
Full independence - Definition (Huang et al. 2009)

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Full independence - Necessary condition

| 1 2 5 3 4 | Each subset of the items maps to a particular subset of the positions |
| 2 1 4 3 5 |
| 1 2 3 4 5 |
| 1 2 3 5 4 |
| 1 2 5 4 3 |
| 2 1 5 3 4 |
Luce model

A ranking $\sigma$ is built by selecting first selecting the preferred item, then among the remaining items, the second preferred and so on. Given $n$ items, each with weight $w_i$

$$P(\sigma) = \prod_{k=1}^{n-1} P_{\{i_k, \ldots, i_n\}}(\sigma(k)) = \prod_{k=1}^{n-1} \frac{w_{\sigma^{-1}(k)}}{\sum_{j=k}^{n} w_{\sigma^{-1}(j)}}$$

L-decomposability induces a conditional independence

$$P(\sigma^{-1}(k) = i_k | \sigma^{-1}(1) = i_1, \ldots, \sigma_{k-1}^{-1}(k-1) = i_{k-1}) = P_{\{i_k, \ldots, i_n\}}(i_k)$$

L-decomposable ranking models

Some Thurstone models and the Mallows model based on Kendall, Hamming or Spearman’s footrule are also L-decomposable
A ranking model $p(\sigma)$ is bi-decomposable iff $p(\sigma)$ and $p(\sigma^{-1})$ are L-decomposable.

A subset of the L-decomposable models.

Mallows-Bradley-Terry
Distance model based on Hamming
Multi-stage models
TL-decomposability (Csiszar 2009)

- A distribution $p(\sigma)$ is TL-decomposable iff $p(\sigma \circ \pi)$ is L-decomposable for every $\pi$
- A subset of the bi-decomposable models
- In other words, for every subset $C$ of the positions the order of these alternatives and the order of the remaining alternatives are independent

TL-decomposition

It can be factored as $p(\sigma) = \prod_{k=1}^{n} c_k(\pi(k))$, $(n - 1)^2$ free parameters

TL-decomposable models

- Mallows-Bradley-Terry
- Distance model based on Hamming
First order marginal independence (Huang et al. 2009)

Necessary (not sufficient) condition for full independence

Given this collection of permutations, the frequency matrix is

\[
\begin{array}{cccc}
1 & 2 & 5 & 3 \\
2 & 1 & 4 & 3 \\
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 5 \\
1 & 2 & 5 & 4 \\
2 & 1 & 5 & 3 \\
4 & 2 & 0 & 0 \\
2 & 4 & 0 & 0 \\
0 & 0 & 2 & 1 \\
0 & 0 & 3 & 2 \\
0 & 0 & 1 & 3 \\
0 & 0 & 1 & 3
\end{array}
\]

Detection: Finding the independent sets is a bi-clustering problem

Approximation quality: What if every time item 2 is in position 1, item 3 is in position 4?

Context: Tracking problems
First order marginal independence (Huang et al. 2009)

Necessary (not sufficient) condition for full independence

Given this collection of permutations, the frequency matrix is

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Detection: Finding the independent sets is a bi-clustering problem

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Context: Tracking problems
Probabilistic Modeling on Rankings

Independence

Riffle shuffle

Given two riffle independent subsets
Set $A = \{1, 2, 3\}$ and $\rho_A(\sigma_A)$
Set $B = \{4, 5, 6\}$ and $\rho_B(\sigma_B)$

relative ordering is maintained

Ranking process of a riffle independent set

$\text{shuffle}(\sigma_A, \sigma_B) = \text{shuffle}((213), (465)) = [(421635), (421653), \ldots]$
Probabilistic Modeling on Rankings

Independence

Riffle shuffle

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Probabilistic Modeling on Rankings
Independence

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Set A={1, 2, 3} and \( p_A(\sigma_A) \)
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Riffle shuffle

Ranking process of a riffle independent set

\[
\text{shuffle}(\sigma_A, \sigma_B) = \text{shuffle}((213), (465)) = \[(421635), (421653), \ldots \]
\]
Distribution over the shuffling, $p(\text{shuffle}(A, B))$

$p((RLLRLR)) = 0.1$
$p((RLLRRL)) = 0.12$
$\vdots$
**Definition**

Distribution $p$ is said to be riffle independent if

$$p(\sigma) = p(\text{shuffle}(A, B)) \cdot p_A(\sigma_A) \cdot p_B(\sigma_B)$$

**Simplifications**

- Different settings of the interleaving distribution lead to simpler models.
- A generalization of the full independent model.

**To take home**

- Natural criterion in the ranking domain.
- First generate the rankings on the independent subsets and then shuffle those rankings to obtain the complete one.
Summary

- **L-decomposability**: the probability of selecting an item at each stage does not depend on the order of the already selected items.

- **First order independence**: a subset of items $A$ always map into a subset of the positions $X$ ($|A| = |X|$).

- **Riffle independence**: given 2 riffle independent subsets $A$ and $B$ and items $a_1, a_2 \in A$ and $b \in B$ knowing that $a_1$ is ranked before $a_2$ does not give any info about where $b$ is ranked.
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Application of ranking has changed in history, from psychology to machine learning.

So has the size and amount of data, from small datasets to huge amount of data.

**Kinds of data**

- Partial ranking: not every items has received a rank, top 5 ranking
- Rank with ties: a set of items is preferred to another set but there is not order within the items in the set
- Implicit data: the acts of users are known (buying film A or visiting B web page) but not their opinion about it, there is no rating or ranking no negative feedback
German sample (Croon, 1988)

Small database of the permutations of 4 elements cited in (Bückenholt, U. 2002)

Idea (Fligner et al. 1986)

98 college students where asked to rank five words, namely thought, play, theory, dream, and attention regarding its association with the word idea

Paper that cite it are (Gupta et al. 2002)
### Cars (Bücken Holt, 2002)

279 Spanish college students were asked to rank four compact cars according to their purchase preferences. The authors provide the ranking patterns’ observed frequencies in this sample.

### APA (Diaconis, 1988)

- The American Psychological Association dataset includes 15449 ballots of the election of the president in 1980, 5738 of which are complete rankings, in which the candidates are ranked from most to least favorite.
- Candidates belong to 3 different schools: clinical, research and community.
- It is very frequently used (Huang et. al 2010).
Sushi (Kamishima et al. 2010)
This information collected via web comprises user information as well as their preferences in sushi, including
- 1025 partial rankings on 100 different kinds of sushi
- 1025 full rankings on 10 different kinds of sushi (items are ranked by every ranker)
Cited by (Huang et. al 2010)

Irish Election (Gormley, et al. (2006))
Results of the election for the Irish House of Parliament election dataset from the Meath constituency in Ireland. It used also in (Huang et. al 2010)
Paired comparisons (Hunter, 2003)
- NFL dataset, results of the 1997 NFL season
- NASCAR dataset, results of the 2002 car season, 36 races and 83 divers took part

Jester (Goldberg et al. 2001)
- 4.1 Million continuous ratings (-10.00 to +10.00) of 100 jokes from 73421 users
- Appears in (Meila et al. 2010)
Both are offline

### Netflix (Bennett, et al. 07)

Movie recommender system. In 2009 1000000 dollar prize was given to Bellkor’s programatic chaos for besting Netflix’s recommender offline for privacy reasons.

### WikiLens

WikiLens was a generalized collaborative recommender system that allowed its community to define item types (movie, book album, restaurant, web) and categories for each type, and then rate and get recommendations for items.
Book crossing (Ziegler, 2005)

Collected from the Book-Crossing community. Contains 278.858 users (with demographic information) providing 1.149.780 ratings (explicit / implicit) about 271.379 books
MovieLens

Rankings with-ties

- MovieLens 100k: 100000 ratings (1-5) from 1000 users on 1700 movies.
- MovieLens 1M: 1 million ratings from 6000 users on 4000 movies.
- MovieLens 10M: 10 million ratings and 100000 tag applications applied to 10000 movies by 72000 users. In the dataset, the movies are linked to movie review systems. Each movie does have its IMDb and RT identifiers, English and Spanish titles, picture URLs, genres, directors, actors (ordered by “popularity”), RT audience’ and experts’ ratings and scores, countries, and filming locations.
HetRec 2011 conference

Highly sparse databases

- **movielens 2k**: extension of MovieLens10M dataset, which contains personal ratings and tags about movies.
- **delicious 2k**: implicit data from Delicious social bookmarking system. Each of the 1867 users has bookmarks, tag assignments, i.e. tuples [user, tag, bookmark], and contact relations within the dataset social network. Each bookmark has a title and URL.
- **lastfm 2k**: implicit dataset in which each user has a list of most listened music artists, tag assignments, i.e. tuples [user, tag, artist], and friend relations within the dataset social network. Each artist has a Last.fm URL and a picture URL.
Non parametric models

- The SnOB software provides general functions or permutations such as Fast Fourier transform.
- The props toolbox (based on SnOB) provides some efficient inference algorithms for distributions over permutations and examples as described in the paper (Huang et al. 2009).
Mallows model

pyMallows is a set of Python routines for fitting and simulating from a generalized Mallows model based on Kendall’s-\(\tau\) distance. Learning algorithms implemented for both full and partial rankings.
Paired comparisons

- *prefmod* fits and simulates data as pairs comparisons, Bradley-Terry models and others, and their extension. The package includes datasets for testing.

- *BradleyTerry2, eba, psychotree* are other R packages for paired comparisons.

- *MMBT* is a Matlab package for the estimation of the Bradley-Terry model and its extension to multi comparison case.
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Conclusions

- Many processes are based on orderings and ranking.
- New technologies allow to store thousands of partial rankings of thousand of items.
- New probabilistic approaches need to be developed to deal with them.
- Two ways:
  - To adapt well-known probabilistic models
  - To create new specific models for the new kind of data